

# GROUP THEORETIC DIMENSION OF STATIONARY SYMMETRIC $\alpha$ -STABLE RANDOM FIELDS

ARIJIT CHAKRABARTY AND PARTHANIL ROY

**ABSTRACT.** The growth rate of the partial maximum of a stationary stable process was first studied in the works of Samorodnitsky (2004a,b), where it was established, based on the seminal works of Rosiński (1995, 2000), that the growth rate is connected to the ergodic theoretic properties of the flow that generates the process. The results were generalized to the case of stable random fields indexed by  $\mathbb{Z}^d$  in Roy and Samorodnitsky (2008), where properties of the group of nonsingular transformations generating the stable process were studied as an attempt to understand the growth rate of the partial maximum process. This work generalizes this connection between stable random fields and group theory to the continuous parameter case, that is, to the fields indexed by  $\mathbb{R}^d$ .

## 1. INTRODUCTION

This paper investigates the growth rate of the partial maxima of stationary symmetric stable non-Gaussian random fields indexed by  $\mathbb{R}^d$ . Let  $\mathbf{X} := \{X_t : t \in \mathbb{R}^d\}$  be a measurable and stationary symmetric  $\alpha$ -stable (S $\alpha$ S) random field with  $0 < \alpha < 2$ . This means that for all  $c_1, c_2, \dots, c_k \in \mathbb{R}$ , and,  $t, t_1, t_2, \dots, t_k \in \mathbb{R}^d$ ,  $\sum_{j=1}^k c_j X_{t_j+t}$  follows a symmetric  $\alpha$ -stable distribution that does not depend on  $t$ . For further reference on S $\alpha$ S distributions and processes, the reader is referred to Samorodnitsky and Taqqu (1994).

Stationarity ensures that the law of the random field  $\mathbf{X}$  is invariant under the shift action of the group  $\mathbb{R}^d$  on the index-parameter of the field. This group action, when viewed in the space of integral representations with respect to S $\alpha$ S random measures (see Samorodnitsky and Taqqu (1994)), is not necessarily invariant but remains nonsingular. This was established in the seminal works of Rosiński (1995) (for  $d = 1$ ) and Rosiński (2000) (for  $d > 1$ ).

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The nonsingular group action obtained in Rosiński (1995, 2000) plays a very important role in the behavior of extremes of stationary S $\alpha$ S random fields. This connection was first explored in the one-dimensional case in Samorodnitsky (2004a,b), which was later generalized for any  $d \geq 2$  in Roy and Samorodnitsky (2008) and Roy (2010b). These works dealt with the partial maxima process of stationary S $\alpha$ S random fields when the index parameter runs over a  $d$ -dimensional hypercube of length increasing to infinity. The rate of growth of this maxima process was exactly calculated when the underlying group action is not conservative. In the case of conservative actions, however, only an upper estimate on the rate of growth could be given in general.

In some discrete multiparameter cases, using the group theoretic structures of the underlying group action, a better estimate on the rate (sometimes the exact rate) of growth of the partial maxima sequence has been given in Section 5 of Roy and Samorodnitsky (2008). The current paper extends this connection with group theory to the continuous parameter case by approximating the random field  $\mathbf{X}$  by its discrete parameter skeletons.

This paper is organized as follows. In Section 2, we present some preliminaries. Section 3 contains the main result of this paper and a couple of examples. The main result is then proved in Section 4 based on a bunch of ergodic theoretic Lemmas, whose proofs are presented in the Appendix.

## 2. PRELIMINARIES

As mentioned above, nonsingular group actions play a significant role in the study of stationary stable random fields and hence, we start with a brief introduction to such actions. Let  $(G, +)$  be a topological group with identity element 0 and Borel  $\sigma$ -field  $\mathcal{G}$ , and  $(S, \mathcal{S}, \mu)$  be a  $\sigma$ -finite measure space. A collection of measurable maps  $\{\phi_t\}_{t \in G}$  on  $S$  is called a nonsingular  $G$ -action on  $S$  if there exists  $S' \in \mathcal{S}$  with  $\mu(S \setminus S') = 0$  such that

- (1)  $(t, s) \mapsto \phi_t(s)$  is a measurable map from  $(G \times S', \mathcal{G} \otimes \mathcal{S}')$  to  $(S', \mathcal{S}')$  (here  $\mathcal{S}'$  is the restriction of the  $\sigma$ -field  $\mathcal{S}$  on  $S'$ ),
- (2)  $\phi_0(s) = s$  for all  $s \in S'$ ,
- (3)  $\phi_{t_1+t_2}(s) = \phi_{t_1} \circ \phi_{t_2}(s)$  for all  $t_1, t_2 \in G$  and  $s \in S'$ , and
- (4)  $\mu \circ \phi_t \sim \mu$  for all  $t \in G$  (here “ $\sim$ ” denotes equivalence of measures).

See, for instance, Aaronson (1997), Krengel (1985), Varadarajan (1970) and Zimmer (1984) for detailed discussions on nonsingular (also known as quasi-invariant) group actions.

If  $G$  is countable then  $W \in \mathcal{S}$  is called a wandering set if  $\{\phi_t(W) : t \in G\}$  is a pairwise disjoint collection and  $\{\phi_t\}_{t \in G}$  is called conservative if it does not admit any wandering set of positive  $\mu$ -measure. On the other hand, for  $G = \mathbb{R}^d$ , it can be shown, based on a result of Kolodyński and Rosiński (2003), that if the restriction  $\{\phi_t\}_{t \in \mathbb{Z}^d}$  is conservative then so are  $\{\phi_t\}_{t \in 2^{-i}\mathbb{Z}^d}$  for all  $i = 1, 2, \dots$ ; see Proposition 2.1 in Roy (2010b). Therefore, the group action  $\{\phi_t\}_{t \in \mathbb{R}^d}$  can be defined to be conservative in this case.

We now present the connection between structures of stationary S $\alpha$ S random fields and nonsingular group actions. It is known that any measurable S $\alpha$ S random field  $\mathbf{X} = \{X_t : t \in \mathbb{R}^d\}$  (not necessarily stationary) has an integral representation given by

$$(2.1) \quad \{X_t : t \in \mathbb{R}^d\} \stackrel{d}{=} \left\{ \int_S f_t(s) \tilde{M}(ds) : t \in \mathbb{R}^d \right\},$$

where  $\tilde{M}$  is a S $\alpha$ S random measure on some standard Borel space  $(S, \mathcal{S})$  with a  $\sigma$ -finite control measure  $\mu$ ,  $f_t \in L^\alpha(S, \mu)$  for all  $t \in \mathbb{R}^d$  and  $(t, s) \mapsto f_t(s)$  is a jointly measurable map; see Samorodnitsky and Taqqu (1994) and Rosiński and Woyczyński (1986). Without loss of generality we can assume that the full support condition

$$\text{support} \left\{ f_t : t \in \mathbb{R}^d \right\} = S$$

holds for any integral representation  $\{f_t\}$  of  $\mathbf{X}$ .

The structure of stationary S $\alpha$ S random fields has been studied in Rosiński (1995, 2000). In those works, it has been shown that the functions  $f_t$  in (2.1) can be chosen to be of the form

$$(2.2) \quad f_t(s) = c_t(s) \left( \frac{d\mu \circ \phi_t}{d\mu}(s) \right)^{1/\alpha} f \circ \phi_t(s), \quad t \in \mathbb{R}^d,$$

where  $f \in L^\alpha(S, \mu)$ ,  $\{\phi_t : t \in \mathbb{R}^d\}$  is a nonsingular group action of the group  $\mathbb{R}^d$  on  $S$  and  $\{c_t : t \in \mathbb{R}^d\}$  is a cocycle, *i.e.*,  $(t, s) \mapsto c_t(s)$  is a jointly measurable function from  $\mathbb{R}^d \times S$  to  $\{-1, 1\}$  such that for all  $u, v \in \mathbb{R}^d$ ,  $c_{u+v}(s) = c_v(s)c_u(\phi_v(s))$  for  $\mu$ -a.a.  $s \in S$ . Conversely,  $\{X_t : t \in \mathbb{R}^d\}$  defined by (2.1) and (2.2) is a stationary S $\alpha$ S random field.

**Remark 2.1.** In fact, Rosiński (1995, 2000) established that every minimal representation (see Hardin Jr. (1982)) of  $\mathbf{X}$  is of the form (2.2). Although this Rosiński representation may not be unique, it has been established based on a rigidity result of  $L^\alpha$  spaces (due to Hardin Jr. (1981)) that if in one Rosiński representation of  $\mathbf{X}$ , the underlying group action is conservative then so is the action in all Rosiński representations; see Rosiński (1995, 2000), Roy and Samorodnitsky (2008) and Roy (2010b). In other words, the spaces of stationary measurable S $\alpha$ S random fields generated by conservative and nonconservative actions are disjoint.

Now we turn our attention to the extremes of stationary measurable S $\alpha$ S random fields. To this end, we assume further that  $\mathbf{X}$  is locally bounded (see, for example, Samorodnitsky and Taqqu (1994) for sufficient conditions for local boundedness of  $\mathbf{X}$ ). Since  $\mathbf{X}$  is stationary and measurable, it is continuous in probability by Proposition 3.1 in Roy (2010b). Therefore, as in the one-dimensional case in Samorodnitsky (2004b), we can take its separable version and define (avoiding the usual measurability problems)

the finite-valued maxima process

$$(2.3) \quad M_t := \sup_{s \in [-t\mathbf{1}, t\mathbf{1}] \cap \Gamma} |X_s|, \quad t \geq 0,$$

where  $\Gamma := \cup_{n=0}^{\infty} \Gamma_n$  with  $\Gamma_n := 2^{-n}\mathbb{Z}^d$ ,  $n \geq 0$  and  $[u, v] := \{s \in \mathbb{R}^d : u \leq s \leq v\}$  (the inequality should be interpreted componentwise).

As mentioned earlier, the rate of growth of  $M_t$  was studied in Samorodnitsky (2004b) and Roy (2010b), where it was established that as  $t \rightarrow \infty$ ,

$$t^{-d/\alpha} M_t \Rightarrow \begin{cases} 0 & \text{if } \{\phi_t\}_{t \in \mathbb{R}^d} \text{ is conservative,} \\ \text{Fréchet distribution} & \text{if } \{\phi_t\}_{t \in \mathbb{R}^d} \text{ is not conservative,} \end{cases}$$

where  $\{\phi_t\}_{t \in \mathbb{R}^d}$  is as in (2.2). Note that the above facts are in agreement with Remark 2.1. This phase transition can be argued to be a transition from longer to shorter memory as described in Samorodnitsky (2004a). In particular, this means that only an upper bound can be given on the rate of growth of the partial maxima process  $M_t$  when the underlying action is conservative.

For the discrete parameter case when the underlying  $\mathbb{Z}^d$ -action is conservative, depending on the group theoretic properties of the underlying action, a better estimate of the rate of growth of the partial maxima sequence

$$M'_n := \max_{s \in [-n\mathbf{1}, n\mathbf{1}] \cap \mathbb{Z}^d} |X_s|, \quad n \geq 0$$

was given in Roy and Samorodnitsky (2008). This work had the following key idea: instead of looking at  $\{\phi_t\}$  as a  $\mathbb{Z}^d$ -action, look at it as a group

$$A = \{\phi_v : v \in \mathbb{Z}^d\}$$

of nonsingular transformations on  $S$  in order to remove the redundancy in the action. This group  $A$  happens to be a quotient group of  $\mathbb{Z}^d$  and hence, by the structure theorem of finitely generated Abelian groups (see, for example, Lang (2002)), can be decomposed as a direct sum of two subgroups, one of which is a free Abelian group  $\bar{F}$  and the other is a finite group  $\bar{N}$ . The subgroup  $\bar{N}$  corresponds to the cycles in the action and  $\bar{F}$  being a free Abelian group has an isomorphic copy  $F$  sitting inside  $\mathbb{Z}^d$  which can be thought of as the effective index set of the random field  $\{X_t\}$ . See Section 3 below for the details.

Roy and Samorodnitsky (2008) showed, using a counting argument based on De Loera (2005), that the restriction of the underlying action on  $F$  reveals extra information on the rate of growth of  $M'_n$  if  $p := \text{rank}(F) < d$ . More specifically, as  $n \rightarrow \infty$ ,

$$n^{-p/\alpha} M'_n \Rightarrow \begin{cases} 0 & \text{if } \{\phi_t\}_{t \in F} \text{ is conservative,} \\ \text{Fréchet distribution} & \text{if } \{\phi_t\}_{t \in F} \text{ is not conservative.} \end{cases}$$

In the above set up,  $p$  can be regarded as the effective dimension of the field. See also Roy (2010a), which investigates a deeper connection between effective dimension and extremes of discrete parameter stable fields.

This concept of effective dimension has not been extended to the continuous parameter case so far because in that case the group  $\{\phi_v : v \in \mathbb{R}^d\}$  of nonsingular transformations is not finitely generated and therefore the structure theorem of finitely generated Abelian groups cannot be used anymore. In this work, we remove this obstacle by observing that the effective dimensions of the discrete parameter subfields  $\{X_t\}_{t \in 2^{-i}\mathbb{Z}^d}$ ,  $i = 0, 1, \dots$  are all equal (see Proposition 3.1 below), which can be defined as the group theoretic dimension of  $\mathbf{X}$ . Our main goal is to investigate the relationship between this group theoretic dimension and the rate of growth of  $M_t$ .

### 3. THE MAIN RESULT

In this section, we state the main theorem of this paper. As mentioned above, we would like to generalize the results given in the Section 5 of Roy and Samorodnitsky (2008) to the continuous parameter case. More specifically, we would like to extend Theorem 5.4 therein. In order to do that, we follow the notations in Roy and Samorodnitsky (2008) and introduce the notion of group theoretic dimension for the continuous parameter stationary S $\alpha$ S random field  $\mathbf{X}$ .

For all  $i \in \{0, 1, 2, \dots\}$ , we define

$$A_i := \{\phi_u : u \in \Gamma_i\},$$

where  $\Gamma_i := 2^{-i}\mathbb{Z}^d$  as defined in Section 2. By first isomorphism theorem of groups (see Lang (2002)), it follows that  $A_i \simeq \Gamma_i/K_i$ , where

$$K_i := \{u \in \Gamma_i : \phi_u = \mathbf{1}_S\}$$

( $\mathbf{1}_S$  denotes the identity map on  $S$ ) is the kernel of the group homomorphism

$$\Phi_i : \Gamma_i \rightarrow A_i$$

defined by  $\Phi_i(u) = \phi_u$ ,  $u \in \Gamma_i$ . In particular, as in Roy and Samorodnitsky (2008),  $A_i$  is a finitely generated Abelian group and hence by appealing to the structure theorem of such groups (see Theorem 8.5 in Chapter 1 of Lang (2002)), we get  $\bar{F}_i, \bar{N}_i \subset A_i$  such that  $\bar{F}_i$  is a free Abelian group of rank  $p_i \leq d$ ,  $\bar{N}_i$  is a finite group,

$$\bar{F}_i \cap \bar{N}_i = \{0\},$$

and

$$A_i = \bar{F}_i + \bar{N}_i.$$

Since  $\bar{F}_i$  is free, there exists an injective group homomorphism  $\Psi_i : \bar{F}_i \rightarrow \Gamma_i$  so that the following diagram commutes:

$$(3.1) \quad \begin{array}{ccc} \Gamma_i & \xrightarrow{\Phi_i} & A_i \\ \uparrow \text{inclusion} & & \uparrow \text{inclusion} \\ F_i & \xleftarrow{\Psi_i} & \bar{F}_i \end{array}$$

where  $F_i := \Psi_i(\bar{F}_i)$  is a free subgroup of  $\Gamma_i$  of rank  $p_i \leq d$ .

Following Roy and Samorodnitsky (2008),  $p_i$  can be regarded as the effective dimension of the subfield  $\{X_t\}_{t \in \Gamma_i}$ . Recall that  $p_i$  determines the rate of growth of the subfield. The first result of this paper shows that the  $p_i$ 's are all equal.

**Proposition 3.1.** *For all  $i \geq 0$ ,  $p_i = p_{i+1}$ .*

*Proof.* Fix  $i \geq 0$ . Since  $K_i$  is a subgroup of  $\mathbb{Z}^d$ , it is a free Abelian group. Let  $q_i := \text{rank}(K_i)$ . We start by showing that

$$(3.2) \quad p_i + q_i = d.$$

Note that  $F_i$  and  $K_i$  are subgroups of  $\Gamma_i$  of rank  $p_i$  and  $q_i$  respectively, with  $F_i \cap K_i = \{0\}$ . Thus, it is immediate that  $p_i + q_i \leq d$ . To show the other inequality, it suffices to prove that

$$(3.3) \quad r\Gamma_i \subset F_i + K_i,$$

where  $r := |\bar{N}_i|$ . To that end, fix  $u \in \Gamma_i$  and notice that  $\Phi_i(u)$  being an element of  $\Gamma_i/K_i$ , can be written as

$$\Phi_i(u) = \bar{v} + y$$

where  $\bar{v} \in \bar{F}_i$  and  $y \in \bar{N}_i$ . Since  $|\bar{N}_i| = r$ , it follows that  $ry = 0$ , and hence

$$\Phi_i(ru) = r\bar{v}.$$

Define  $v := \Psi_i(r\bar{v})$ . Since the diagram in (3.1) commutes, it follows that

$$\Phi_i(v) = r\bar{v} = \Phi_i(ru).$$

Thus,  $ru - v \in \text{Ker}(\Phi_i) = K_i$ , which shows (3.3) and consequently proves (3.2). In view of (3.2), it suffices to show that  $q_i = q_{i+1}$  and that follows trivially because  $2K_{i+1} \subset K_i \subset K_{i+1}$ .  $\square$

Based on the preceding result, we denote

$$p := p_0 = p_1 = \dots \leq d$$

and define it to be the group theoretic dimension of the random field  $\mathbf{X}$ . We shall assume throughout the paper that  $p \geq 1$  (see Remark 5.5 of Roy and Samorodnitsky (2008)). As in the discrete parameter case,  $p$  has information on the rate of growth of the partial maxima (2.3) as described below in Theorem 3.2, which extends Theorem 5.4 of Roy and Samorodnitsky (2008) to the continuous parameter case and is the main result of this paper.

**Theorem 3.2.** *(i) If the group action  $\{\phi_t : t \in F_0\}$  is not conservative, then*

$$(3.4) \quad t^{-p/\alpha} M_t \Longrightarrow K Z_\alpha$$

as  $t \rightarrow \infty$ , where  $K \in (0, \infty)$  is a constant and  $Z_\alpha$  is the standard Fréchet-type extreme value random variable with c.d.f.

$$P(Z_\alpha \leq z) = \exp(-z^{-\alpha})$$

for  $z > 0$ .

(ii) If the group action  $\{\phi_t : t \in F_0\}$  is conservative, then

$$(3.5) \quad t^{-p/\alpha} M_t \xrightarrow{P} 0$$

as  $t \rightarrow \infty$ .

**Remark 3.3.** The constant  $K$  in (3.4) can be written as

$$K := C_\alpha^{1/\alpha} K_X,$$

where  $C_\alpha$  is the stable tail constant given by

$$C_\alpha = \left( \int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} & \text{if } \alpha = 1, \end{cases}$$

and  $K_X$  is the constant that equals the limit in (4.1) below.

**Remark 3.4.** As described in Samorodnitsky (2004a), the maxima process grows in a smaller rate when  $\{X_t\}$  has stronger dependence due to the conservativity of the underlying action. Therefore, stronger conservativity of the action yields a smaller value of  $p$  (because of a bigger  $K_0$ ) and hence smaller rate of growth of  $M_t$  and this is manifested in Theorem 3.2.

We shall prove Theorem 3.2 in the next section. We first illustrate it with the following continuous parameter analogue of Example 6.2 in Roy and Samorodnitsky (2008).

**Example 3.5.** Suppose

$$U := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$$

is the unit circle. We take  $d = 3$ , and define the  $\mathbb{R}^3$ -action  $\{\phi_{(x,y,z)}\}$  on  $S = \mathbb{R} \times U$  as

$$\phi_{(x,y,z)}(s, \zeta) = (s + x - y, \zeta e^{i2\pi z}).$$

Clearly, this action preserves the measure  $\mu$  on  $S$  defined as the product of the Lebesgue measure on  $\mathbb{R}$  and the Haar probability measure on  $U$ . In parallel to Example 6.2 in Roy and Samorodnitsky (2008), we can take any  $f \in L^\alpha(S, \mu)$  and define a stationary S $\alpha$ S random field  $\{X_{(x,y,z)}\}$  by

$$X_{(x,y,z)} = \int_{\mathbb{R} \times U} f(\phi_{(x,y,z)}(s, \zeta)) \, d\tilde{M}(s, \zeta),$$

where  $\tilde{M}$  is a S $\alpha$ S random measure on  $\mathbb{R} \times U$  with control measure  $\mu$ .

For all  $i = 0, 1, 2, \dots$ ,  $K_i = \{(x, y, z) \in \Gamma_i : x = y, z \in \mathbb{Z}\}$  and therefore following the calculations in Example 6.2 of Roy and Samorodnitsky (2008), we get  $A_i \simeq 2^{-i}\mathbb{Z} \times (\mathbb{Z}/2^i\mathbb{Z})$  and  $F_i = 2^{-i}\mathbb{Z} \times \{0\} \times \{0\}$ . In particular,  $p = 1$  and  $\{\phi_t\}_{t \in F_0}$  is not conservative since  $W := (0, 1) \times U$  is a wandering set of positive  $\mu$ -measure. Therefore, Theorem 3.2 yields that  $t^{-1/\alpha} M_t$  converges to a Fréchet type extreme value random variable.

**Remark 3.6.** Theorem 3.2 above is expected to give better results than Theorem 4.1 of Roy (2010b) when the underlying action is conservative. For example, in Example 3.5, Theorem 4.1 of Roy (2010b) would just yield  $M_t = o_p(t^{3/\alpha})$  as opposed to  $M_t = O_p(t^{1/\alpha})$ . However, this is not always the case as shown in the following example.

**Example 3.7.** Consider the continuous parameter analogue of Example 6.3 (based on an action suggested by M. G. Nadkarni) in Roy and Samorodnitsky (2008):  $S = \mathbb{R}$  endowed with the Lebesgue measure,  $d = 2$ ,  $f := I_{[0,1]}$ , and for all  $(u, v) \in \mathbb{R}^2$ ,

$$\phi_{u,v}(s) := s + u - v\sqrt{2}, s \in \mathbb{R}$$

and  $X_{(u,v)}$  is defined by (2.1) and (2.2). It can be shown that  $M_t = O_p(t^{1/\alpha})$  in this example (see Remark 4.5 below) although both Theorem 3.2 above and Theorem 4.1 of Roy (2010b) would give  $M_t = o_p(t^{2/\alpha})$ .

#### 4. PROOF OF THEOREM 3.2

The rate of growth of the maxima process is determined by that of the deterministic function  $b(T)$  defined as

$$b(T) := \left\{ \int_S \sup_{t \in [-T\mathbf{1}, T\mathbf{1}] \cap \Gamma} |f_t(s)|^\alpha \mu(ds) \right\}^{1/\alpha}.$$

See Samorodnitsky (2004a,b), Roy and Samorodnitsky (2008). We start by studying the growth rate of  $b(T)$  in both the conservative and the non-conservative cases using the discrete parameter approximation of the field and appealing to the results available in Section 5 of Roy and Samorodnitsky (2008).

**Proposition 4.1.** (i) *If the action  $\{\phi_t : t \in F_0\}$  is conservative, then,*

$$\lim_{T \rightarrow \infty} T^{-p/\alpha} b(T) = 0.$$

(ii) *On the other hand, if  $\{\phi_t : t \in F_0\}$  is not conservative, then*

$$(4.1) \quad \lim_{T \rightarrow \infty} T^{-p/\alpha} b(T)$$

*exists, and is finite and positive.*

In order to prove Proposition 4.1, we need the following lemmas, whose proofs are presented in the Appendix.

**Lemma 4.2.** *If for any finitely generated Abelian group  $G$ , the action  $\{\phi_u : u \in G\}$  is conservative, then so is the action  $\{\phi_{ru} : u \in G\}$  for all integers  $r \geq 1$ .*

**Lemma 4.3.** *Suppose that for some integer  $I \geq 0$ , the action  $\{\phi_t : t \in F_I\}$  is conservative. Then, for all  $i \geq 0$ , the action  $\{\phi_t : t \in F_i\}$  is conservative.*



**Lemma 4.4.** *There exist a positive integer  $M$ ,  $u_1, \dots, u_p \in F_0$  and  $v_1, \dots, v_q \in K_0$  so that for all  $n \geq 1$ ,*

$$(4.2) \quad [-n\mathbf{1}, n\mathbf{1}] \cap \Gamma \subset \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma, \right. \\ \left. \alpha_i, \beta_j \in [-Mn, Mn] \cap \mathbb{Q}_{bin} \text{ for all } i, j \right\},$$

where  $q := d - p$  and  $\mathbb{Q}_{bin} := \bigcup_{m=0}^{\infty} 2^{-m}\mathbb{Z}$  denotes the set of binary rationals.

*Proof of Proposition 4.1.* (i) Choose  $M$ ,  $u_1, \dots, u_p$  and  $v_1, \dots, v_q$  so that (4.2) holds. Define

$$E := \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma, \right. \\ \left. \alpha_i \in [0, M] \cap \mathbb{Q}_{bin}, \beta_j \in [0, 1] \cap \mathbb{Q}_{bin} \text{ for all } i, j \right\}$$

and the sets

$$E_m := \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma_m, \right. \\ \left. \alpha_i \in [0, M] \cap 2^{-m}\mathbb{Z}, \beta_j \in [0, 1] \cap 2^{-m}\mathbb{Z} \text{ for all } i, j \right\}, m \geq 0.$$

Clearly,  $E$  is a bounded subset of  $\Gamma$ . By Proposition 10.2.1 and Theorem 10.2.3 of Samorodnitsky and Taqqu (1994) it follows that

$$\int_S \sup_{t \in E} |f_t(x)|^\alpha \mu(dx) < \infty.$$

Fix  $\varepsilon > 0$ . Since  $E_m \uparrow E$ , by the monotone convergence theorem there exists  $m \geq 0$  so that

$$\int_S \max_{t \in E_m} |f_t(x)|^\alpha \mu(dx) \geq \int_S \sup_{t \in E} |f_t(x)|^\alpha \mu(dx) - \varepsilon.$$

Fix such an  $m$ . For  $k_1, \dots, k_p \in \mathbb{Z}$ , define the sets

$$(4.3) \quad \overline{H}(k_1, \dots, k_p) := \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma, \right. \\ \left. \alpha_i \in [k_i M, (k_i + 1)M] \cap \mathbb{Q}_{bin}, \beta_j \in \mathbb{Q}_{bin} \text{ for all } i, j \right\}$$

and

$$H(k_1, \dots, k_p) := \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma_m, \right. \\ \left. \alpha_i \in [k_i M, (k_i + 1)M] \cap 2^{-m}\mathbb{Z}, \beta_j \in 2^{-m}\mathbb{Z} \text{ for all } i, j \right\}.$$

Define for  $n \geq 1$ ,

$$\bar{a}_n := \int_S \sup \left\{ |f_t(x)|^\alpha : t \in \bigcup_{k_1=-n}^{n-1} \dots \bigcup_{k_p=-n}^{n-1} \bar{H}(k_1, \dots, k_p) \right\} \mu(dx)$$

and

$$a_n := \int_S \max \left\{ |f_t(x)|^\alpha : t \in \bigcup_{k_1=-n}^{n-1} \dots \bigcup_{k_p=-n}^{n-1} H(k_1, \dots, k_p) \right\} \mu(dx).$$

Clearly by (4.2),  $b(n)^\alpha \leq \bar{a}_n$ . Note that for  $n \geq 1$ ,

$$\begin{aligned} & \bar{a}_n - a_n \\ & \leq \sum_{k_1=-n}^{n-1} \dots \sum_{k_p=-n}^{n-1} \int_S \left[ \sup_{t \in \bar{H}(k_1, \dots, k_p)} |f_t(x)|^\alpha \right. \\ & \quad \left. - \max_{t \in H(k_1, \dots, k_p)} |f_t(x)|^\alpha \right] \mu(dx). \end{aligned}$$

Note that all the  $H(k_1, \dots, k_p)$ 's are simply translates of each other and therefore using Corollary 4.4.6 of Samorodnitsky and Taqqu (1994), the right hand side of the above inequality equals

$$\begin{aligned} & (2n)^p \int_S \left[ \sup_{t \in \bar{H}(0, \dots, 0)} |f_t(x)|^\alpha - \max_{t \in H(0, \dots, 0)} |f_t(x)|^\alpha \right] \mu(dx) \\ & = (2n)^p \int_S \left[ \sup_{t \in E} |f_t(x)|^\alpha - \max_{t \in E_m} |f_t(x)|^\alpha \right] \mu(dx) \leq (2n)^p \varepsilon. \end{aligned}$$

The above computations put together imply that

$$(4.4) \quad \limsup_{n \rightarrow \infty} n^{-p} b(n)^\alpha \leq 2^p \varepsilon + \limsup_{n \rightarrow \infty} n^{-p} a_n.$$

The next step is to show that

$$(4.5) \quad \lim_{n \rightarrow \infty} n^{-p} a_n = 0.$$

To that end, notice that

$$a_n = \int_S \max \left\{ |f_t(x)|^\alpha : t \in \bigcup_{k_1=-n}^{n-1} \dots \bigcup_{k_p=-n}^{n-1} H'(k_1, \dots, k_p) \right\} \mu(dx),$$

where

$$H'(k_1, \dots, k_p) := \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma_m, \right. \\ \left. \alpha_i \in [k_i M, (k_i + 1)M] \cap 2^{-m}\mathbb{Z}, \beta_j \in [0, 1] \cap 2^{-m}\mathbb{Z} \text{ for all } i, j \right\},$$

for  $k_1, \dots, k_p \in \mathbb{Z}$ . Clearly, there exists a positive integer  $c$  so that

$$\bigcup_{k_1=-n}^{n-1} \dots \bigcup_{k_p=-n}^{n-1} H'(k_1, \dots, k_p) \subset [-cn\mathbf{1}, cn\mathbf{1}] \cap \Gamma_m,$$

for all  $n \geq 1$ . Thus,

$$(4.6) \quad a_n \leq \int_S \max_{t \in [-cn\mathbf{1}, cn\mathbf{1}] \cap \Gamma_m} |f_t(x)|^\alpha \mu(dx).$$

By Lemma 4.3, the group action  $\{\phi_t : t \in F_m\}$  is conservative. An appeal to Proposition 5.1 of Roy and Samorodnitsky (2008) shows that the right hand side of (4.6) is  $o(n^p)$ , and thus proves (4.5). This along with (4.4) and the fact that  $\varepsilon$  there is arbitrary shows that

$$\lim_{n \rightarrow \infty} n^{-p/\alpha} b(n) = 0.$$

The proof follows from here by the observation that

$$(4.7) \quad T^{-p/\alpha} b(T) \leq \left( \frac{\lceil T \rceil}{T} \right)^{p/\alpha} \lceil T \rceil^{-p/\alpha} b(\lceil T \rceil)$$

for all  $T > 0$ .

(ii) As in the proof of Part (i), fix  $M, u_1, \dots, u_p$  and  $v_1, \dots, v_q$  so that (4.2) holds. For  $k_1, \dots, k_p, l_1, \dots, l_q \in \mathbb{Z}$ , define the set

$$\tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q) := \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma, \right. \\ \left. \alpha_i \in [k_i M, (k_i + 1)M] \cap \mathbb{Q}_{bin}, \beta_j \in [l_j M, (l_j + 1)M] \cap \mathbb{Q}_{bin} \text{ for all } i, j \right\}.$$

A restatement of (4.2) is that for  $n \geq 1$ ,

$$(4.8) \quad [-n\mathbf{1}, n\mathbf{1}] \cap \Gamma \subset \bigcup_{-n \leq k_1, \dots, k_p, l_1, \dots, l_q \leq n-1} \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q).$$

Let

$$G_n := \left\{ (k_1, \dots, k_p) \in \mathbb{Z}^p : -n \leq k_i \leq n-1 \text{ for all } i \text{ and } \right. \\ \left. \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q) \cap [-n\mathbf{1}, n\mathbf{1}] \neq \emptyset \text{ for some } l_1, \dots, l_q \in \mathbb{Z} \right\}.$$

Define for  $m, n \geq 1$ ,

$$\tilde{a}_n^{(m)} := \int_S \sup \left\{ |f_t(x)|^\alpha : t \in \bigcup_{(k_1, \dots, k_p) \in G_n} \bigcup_{l_1, \dots, l_q \in \mathbb{Z}} \right. \\ \left. \Gamma_m \cap \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q) \right\} \mu(dx),$$

and

$$\tilde{a}_n := \int_S \sup \left\{ |f_t(x)|^\alpha : t \in \bigcup_{(k_1, \dots, k_p) \in G_n} \bigcup_{l_1, \dots, l_q \in \mathbb{Z}} \right. \\ \left. \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q) \right\} \mu(dx).$$

Let  $K$  be an integer which is no smaller than the diameter (with respect to the  $L^\infty$  norm) of  $\tilde{H}(0, \dots, 0)$ . Let for  $m \geq 1$  and  $T > 0$ ,

$$\tilde{b}_m(T) := \int_S \sup_{t \in [-T\mathbf{1}, T\mathbf{1}] \cap \Gamma_m} |f_t(x)|^\alpha \mu(dx).$$

We claim that the proof will follow if the following can be shown:

$$(4.9) \quad b(n)^\alpha \leq \tilde{a}_n \leq b(n+K)^\alpha \text{ for all } n \geq 1,$$

$$(4.10) \quad \tilde{b}_m(n)^\alpha \leq \tilde{a}_n^{(m)} \leq \tilde{b}_m(n+K)^\alpha \text{ for all } m, n \geq 1,$$

and

$$(4.11) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{-p} (\tilde{a}_n - \tilde{a}_n^{(m)}) = 0.$$

Suppose, for the moment, that the above is true. An appeal to Lemma 4.3 shows that for all  $m \geq 0$ , the group action  $\{\phi_t : t \in F_m\}$  is non-conservative. By Proposition 5.1 in Roy and Samorodnitsky (2008) it follows that

$$\lim_{n \rightarrow \infty} n^{-p} \tilde{b}_m(n)^\alpha = A_m \in (0, \infty) \text{ for all } m \geq 1.$$

This, along with (4.10) yields that

$$(4.12) \quad \lim_{n \rightarrow \infty} n^{-p} \tilde{a}_n^{(m)} = A_m \text{ for all } m \geq 1.$$

We next show that  $A := \sup_{m \geq 1} A_m < \infty$ . To that end, observe that by (4.11), there exists  $m \geq 1$  so that

$$\limsup_{n \rightarrow \infty} n^{-p} (\tilde{a}_n - \tilde{a}_n^{(m)}) \leq 1.$$

Fix such an  $m$ . Thus

$$n^{-p} \left( \tilde{a}_n - \tilde{a}_n^{(m)} \right) \leq 2$$

for  $n$  large enough, and hence

$$\limsup_{n \rightarrow \infty} n^{-p} \tilde{a}_n \leq 2 + A_m.$$

Observe that

$$A \leq \limsup_{n \rightarrow \infty} n^{-p} b(n)^\alpha \leq \limsup_{n \rightarrow \infty} n^{-p} \tilde{a}_n,$$

the second inequality following from (4.9). Thus,  $A < \infty$ . Since for all  $m, n$ ,  $\tilde{b}_m(n) \leq \tilde{b}_{m+1}(n)$ , it follows that  $A_m \leq A_{m+1}$ . Hence,

$$\lim_{m \rightarrow \infty} A_m = A.$$

In view of (4.11) and (4.12), this implies that

$$\lim_{n \rightarrow \infty} n^{-p} \tilde{a}_n = A.$$

As a restatement of (4.9), we have that

$$a_{n-K} \leq b(n)^\alpha \leq \tilde{a}_n$$

for  $n > K$ . Hence,

$$\lim_{n \rightarrow \infty} n^{-p} b(n)^\alpha = A.$$

An observation similar to (4.7) will prove Part (ii) of Proposition 4.1.

We now show (4.9) - (4.11). The first inequality in (4.9) follows trivially from the observation that

$$[-n\mathbf{1}, n\mathbf{1}] \cap \Gamma \subset \bigcup_{(k_1, \dots, k_p) \in G_n} \bigcup_{l_1, \dots, l_q \in \mathbb{Z}} \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q),$$

which in turn is a consequence of (4.8). For the second inequality, it suffices to show that for all

$$t \in \bigcup_{(k_1, \dots, k_p) \in G_n} \bigcup_{l_1, \dots, l_q \in \mathbb{Z}} \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q),$$

there exists  $t' \in \Gamma$  with  $\|t'\| \leq n + K$  so that

$$(4.13) \quad f_t(x) = f_{t'}(x) \text{ for all } x \in S,$$

where  $\|\cdot\|$  denotes the  $L^\infty$  norm on  $\mathbb{R}^d$ . By choice of  $t$ , there exist  $(k_1, \dots, k_p) \in G_n$  and  $l_1, \dots, l_q \in \mathbb{Z}$  so that

$$t \in \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q).$$

Clearly,

$$t = y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j$$

for some  $y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma$ ,  $\alpha_i \in [k_i M, (k_i + 1)M] \cap \mathbb{Q}_{bin}$  and  $\beta_j \in [l_j M, (l_j + 1)M] \cap \mathbb{Q}_{bin}$ . Since  $(k_1, \dots, k_p) \in G_n$ , there exist  $l'_1, \dots, l'_q \in \mathbb{Z}$  so that

$$(4.14) \quad [-n\mathbf{1}, n\mathbf{1}] \cap \tilde{H}(k_1, \dots, k_p, l'_1, \dots, l'_q) \neq \emptyset.$$

Define

$$t' := y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \{\beta_j + (l'_j - l_j)M\} v_j.$$

Clearly, (4.13) holds for this  $t'$ . Thus, for the second inequality in (4.9), it suffices to show that

$$\|t'\| \leq n + K.$$

Since (4.14) holds, there exists

$$s \in [-n\mathbf{1}, n\mathbf{1}] \cap \tilde{H}(k_1, \dots, k_p, l'_1, \dots, l'_q).$$

It is easy to see that the diameters of  $\tilde{H}(k_1, \dots, k_p, l'_1, \dots, l'_q)$  and  $\tilde{H}(0, \dots, 0)$  are the same because one is a translate of the other. Thus, the diameter of the former is bounded by  $K$ . Notice that from the definition of  $t'$ , it is immediate that  $t' \in \tilde{H}(k_1, \dots, k_p, l'_1, \dots, l'_q)$ . Since  $s$  also belongs to that set, it follows that  $\|s - t'\| \leq K$ . Clearly  $\|s\| \leq n$  because  $s \in [-n\mathbf{1}, n\mathbf{1}]$ . This shows that  $\|t'\| \leq n + K$ , and thus proves (4.9). The justification for (4.10) follows by similar arguments.

Finally, we proceed to prove (4.11). Fix  $\varepsilon > 0$ . Fix  $m_0 \geq 0$  such that

$$\int_S \max_{t \in \tilde{H}(0, \dots, 0) \cap \Gamma_{m_0}} |f_t(x)|^\alpha \mu(dx) \geq \int_S \sup_{t \in \tilde{H}(0, \dots, 0)} |f_t(x)|^\alpha \mu(dx) - \varepsilon.$$

Note that for  $m \geq m_0$  and  $n \geq 1$ ,

$$\begin{aligned} & \tilde{a}_n - \tilde{a}_n^{(m)} \\ &= \int_S \left[ \max_{(k_1, \dots, k_p) \in G_n} \sup \left\{ |f_t(x)|^\alpha : t \in \bigcup_{l_1, \dots, l_q \in \mathbb{Z}} \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q) \right\} - \right. \\ & \quad \left. \max_{(k_1, \dots, k_p) \in G_n} \sup \left\{ |f_t(x)|^\alpha : t \in \bigcup_{l_1, \dots, l_q \in \mathbb{Z}} \Gamma_m \cap \tilde{H}(k_1, \dots, k_p, l_1, \dots, l_q) \right\} \right] \mu(dx) \\ &= \int_S \left[ \max_{(k_1, \dots, k_p) \in G_n} \sup \left\{ |f_t(x)|^\alpha : t \in \tilde{H}(k_1, \dots, k_p, 0, \dots, 0) \right\} - \right. \\ & \quad \left. \max_{(k_1, \dots, k_p) \in G_n} \max \left\{ |f_t(x)|^\alpha : t \in \Gamma_m \cap \tilde{H}(k_1, \dots, k_p, 0, \dots, 0) \right\} \right] \mu(dx) \\ &\leq \sum_{(k_1, \dots, k_p) \in G_n} \int_S \left[ \sup_{t \in \tilde{H}(k_1, \dots, k_p, 0, \dots, 0)} |f_t(x)|^\alpha - \right. \\ & \quad \left. \max_{t \in \Gamma_m \cap \tilde{H}(k_1, \dots, k_p, 0, \dots, 0)} |f_t(x)|^\alpha \right] \mu(dx) \\ &\leq (2n)^p \int_S \left[ \sup_{t \in \tilde{H}(0, \dots, 0)} |f_t(x)|^\alpha - \max_{t \in \Gamma_m \cap \tilde{H}(0, \dots, 0)} |f_t(x)|^\alpha \right] \mu(dx) \leq (2n)^p \varepsilon. \end{aligned}$$

Thus,

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{-p} \left( \tilde{a}_n - \tilde{a}_n^{(m)} \right) \leq 2^p \varepsilon.$$

Since,  $\varepsilon$  is arbitrary, this shows (4.11) and completes the proof.  $\square$

Having established Proposition 4.1, we are now ready to prove Theorem 3.2. Let  $K_X$  denote the limit obtained in (4.1). We start by proving (3.4). In view of Proposition 4.1, it suffices to show that

$$(4.15) \quad b_t^{-1} M_t \implies C_\alpha^{1/\alpha} Z_\alpha$$

as  $t \rightarrow \infty$ . For  $t > 0$ , let  $\eta_t$  be a probability measure on  $(S, \mathcal{S})$  with

$$\frac{d\eta_t}{d\mu}(x) = b_t^{-\alpha} \sup_{s \in [-t\mathbf{1}, t\mathbf{1}] \cap \Gamma} |f_s(x)|^\alpha$$

for all  $x \in S$ . Let  $U_j^{(t)}$ ,  $j = 1, 2$ , be independent  $S$ -valued random variables with common law  $\eta_t$ . By Theorem 4.1 in Roy (2010b), in order to establish (4.15), it suffices to show that

$$(4.16) \quad \lim_{t \rightarrow \infty} P \left( \text{for some } s \in [-t\mathbf{1}, t\mathbf{1}] \cap \Gamma, \frac{|f_s(U_j^{(t)})|}{\sup_{u \in [-t\mathbf{1}, t\mathbf{1}] \cap \Gamma} |f_u(U_j^{(t)})|} > \varepsilon, j = 1, 2 \right) = 0$$

for all  $\varepsilon > 0$ . Using arguments given in Samorodnitsky (2004b) (see the proof of (2.14) implies (2.12) therein) and with  $\overline{H}(k_1, \dots, k_p)$  as in (4.3), we have that the probability in (4.16) is bounded by

$$\begin{aligned} & \sum_{k_1 = -\lceil t \rceil}^{\lceil t \rceil - 1} \dots \sum_{k_p = -\lceil t \rceil}^{\lceil t \rceil - 1} P \left( \text{for some } s \in \overline{H}(k_1, \dots, k_p), \right. \\ & \quad \left. \frac{|f_s(U_j^{(t)})|}{\sup_{u \in [-t\mathbf{1}, t\mathbf{1}] \cap \Gamma} |f_u(U_j^{(t)})|} > \varepsilon, j = 1, 2 \right) \\ & \leq (2\lceil t \rceil)^p \varepsilon^{-2\alpha} b_t^{-2\alpha} \left( \int_S \sup_{s \in \overline{H}(0, \dots, 0)} |f_s(x)|^\alpha m(dx) \right)^2 \rightarrow 0 \end{aligned}$$

as  $t \rightarrow \infty$  by (4.1). This shows (4.16) and hence completes the proof of (3.4).

That (3.5) is true, follows easily from Theorem 4.1 of Roy (2010b) and Proposition 4.1 by an argument similar to the proof of (2.7) in Samorodnitsky (2004b).

**Remark 4.5.** It is easy to see that the above proof can be carried over to any finitely generated subgroup  $\Gamma_0$  of  $\mathbb{R}^d$  of rank  $d$  and  $\Gamma_n := 2^{-n}\Gamma_0$  for  $n \geq 1$ . The subgroup  $\Gamma_0$  may be suitably chosen to yield better results in some cases. Consider Example 3.7 once again. Taking  $\Gamma_0 = \{(i\sqrt{2}, j) : i, j \in \mathbb{Z}\}$

and using the estimate obtained in Example 6.3 in Roy and Samorodnitsky (2008), the sharper bound  $M_t = O_p(t^{1/\alpha})$  follows although choosing  $\Gamma_0 = \mathbb{Z}^2$  would just give  $M_t = o_p(t^{2/\alpha})$ .

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#### APPENDIX A. PROOFS OF THE LEMMAS

**A.1. Proof of Lemma 4.2.** If possible, let  $A$  be a wandering set of positive measure for  $\{\phi_{ru} : u \in G\}$ . By Proposition 2.2 in Roy and Samorodnitsky (2008), it follows that

$$(A.1) \quad \sum_{t \in G} \mathbf{1}_A \circ \phi_t = \infty \text{ a.e. on } A.$$

Suppose that  $G$  has a generating set of size  $k$ . Since  $A$  is a wandering set for  $\{\phi_{ru} : u \in G\}$ , by arguments similar to those in the proof of Theorem 3.4, page 18 in Krengel (1985), it follows that

$$(A.2) \quad \sum_{t \in G} \mathbf{1}_A \circ \phi_t \leq r^k.$$

Clearly, (A.1) and (A.2) contradict each other. This completes the proof.

**A.2. Proof of Lemma 4.3.** We first show that for  $i \geq I + 1$ , the action  $\{\phi_t : t \in F_i\}$  is conservative. Without loss of generality we can and do assume that  $I = 0$ , because otherwise, the elements of  $\Gamma_I$  can be relabeled to become  $\mathbb{Z}^d$ . We show that  $\{\phi_t : t \in F_1\}$  is a conservative action. By similar arguments, the result will follow for all  $i$ . All that needs to be shown is that given  $A \in \mathcal{S}$  with  $\mu(A) > 0$  there exist  $s, t \in F_1$  with  $s \neq t$  so that

$$(A.3) \quad \mu(\phi_s(A) \cap \phi_t(A)) > 0.$$

Fix such an  $A$ . Let  $r := |\bar{N}_1| \geq 1$ . By Lemma 4.2 it follows that  $\{\phi_{rt} : t \in F_0\}$  is a conservative action. Thus, there exist  $s', t' \in F_0$  with  $s' \neq t'$  so that

$$(A.4) \quad \mu(\phi_{rs'}(A) \cap \phi_{rt'}(A)) > 0.$$

Since  $s' \in F_0 \subset \Gamma_0 \subset \Gamma_1$ ,  $\Phi_1(s') \in \Gamma_1/K_1$ . Since

$$\Gamma_1/K_1 = \bar{F}_1 + \bar{N}_1,$$

there exist  $\bar{s} \in \bar{F}_1$  and  $\bar{y} \in \bar{N}_1$  so that  $\Phi_1(s') = \bar{s} + \bar{y}$ . Using the fact that  $r\bar{y} = 0$ , it follows that

$$\Phi_1(rs') = r\bar{s}.$$

Define

$$s := \Psi_1(r\bar{s}).$$

Clearly,  $s \in F_1$ . Using (3.1) with  $i = 1$  yields that

$$\Phi_1(s) = r\bar{s} = \Phi_1(rs').$$



Thus,  $s - rs' \in K_1$  and hence

$$\phi_s = \phi_{rs'}.$$

By similar arguments it follows that  $\Phi_1(rt') \in \overline{F}_1$ , and hence one can define

$$t := \Psi_1 \circ \Phi_1(rt').$$

Once again, the same arguments as above will show that

$$\phi_t = \phi_{rt'}.$$

An appeal to (A.4) shows that (A.3) holds with this choice of  $s$  and  $t$ . To complete the proof, all that needs to be checked is that  $s \neq t$ . This is immediate because if  $s$  and  $t$  are equal then so are  $\Phi_1(rs')$  and  $\Phi_1(rt')$  and therefore,  $rs' - rt' \in K_1 \cap F_0 \subseteq K_0 \cap F_0 = \{0\}$  yielding  $s' = t'$ , which is a contradiction. This proves the fact that for  $i \geq I+1$ , the action  $\{\phi_t : t \in F_i\}$  is conservative.

We now show that the action  $\{\phi_t : t \in F_i\}$  is conservative for  $i \leq I$ . Since  $\{\phi_t : t \in F_I\}$  is a conservative action, by (5.1) of Roy and Samorodnitsky (2008), it follows that

$$\lim_{n \rightarrow \infty} n^{-p} \int_S \sup_{t \in [-n\mathbf{1}, n\mathbf{1}] \cap \Gamma_I} |f_t(x)|^\alpha \mu(dx) = 0.$$

An immediate consequence of this is that

$$\lim_{n \rightarrow \infty} n^{-p} \int_S \sup_{t \in [-n\mathbf{1}, n\mathbf{1}] \cap \Gamma_i} |f_t(x)|^\alpha \mu(dx) = 0,$$

for all  $0 \leq i < I$ . From here, it follows that  $\{\phi_t : t \in F_i\}$  is a conservative action, for otherwise, (5.2) of Roy and Samorodnitsky (2008) would be contradicted, which can easily be seen to hold for non-conservative actions as well (not only for dissipative actions) by decomposing the action into conservative and dissipative parts (see Aaronson (1997)). This proves Lemma 4.3.

**A.3. Proof of Lemma 4.4.** Lemma 5.2 of Roy and Samorodnitsky (2008) shows the existence of a positive integer  $M'$ ,  $x_1, \dots, x_l \in \mathbb{Z}^d$ ,  $u_1, \dots, u_p \in F_0$  and  $v_1, \dots, v_q \in K_0$  so that for all  $n \geq M'$ ,

$$(A.5) \quad [-n\mathbf{1}, n\mathbf{1}] \cap \mathbb{Z}^d \subset \bigcup_{k=1}^l \left\{ x_k + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : \right. \\ \left. \alpha_i, \beta_j \in [-M'n, M'n] \cap \mathbb{Z} \text{ for all } i, j \right\}.$$

Define  $M := M' \vee \|x_1\| \vee \|x_2\| \cdots \vee \|x_l\|$ . In order to establish (4.2), it is enough to show that for all  $m \geq 0$ ,

$$[-n\mathbf{1}, n\mathbf{1}] \cap \Gamma_m \subset \left\{ y + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j : y \in [-M\mathbf{1}, M\mathbf{1}] \cap \Gamma_m, \right. \\ \left. \alpha_i, \beta_j \in [-Mn, Mn] \cap 2^{-m}\mathbb{Z} \text{ for all } i, j \right\},$$

which follows by induction on  $m \geq 0$  from (A.5) and the observation that

$$[-n\mathbf{1}, n\mathbf{1}] \cap \Gamma_{m+1} \subset \frac{1}{2} \{([-n\mathbf{1}, n\mathbf{1}] \cap \Gamma_m) + ([-n\mathbf{1}, n\mathbf{1}] \cap \Gamma_m)\}.$$

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STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, DELHI, INDIA  
*E-mail address:* `arijit@isid.ac.in`

STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, KOLKATA, INDIA  
*E-mail address:* `parthanil.roy@gmail.com`